ACADEMIC PRESS

Available online at www.sciencedirect.com
SCIENCE DIRECT

Journal of Sound and Vibration 263 (2003) 535–548

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

A new approach with piecewise-constant arguments to approximate and numerical solutions of oscillatory problems

L. Dai[a,*], M.C. Singh[b]

[a] *Industrial Systems Engineering, University of Regina, 3737 Wascana Parkway, Regina, Sask., Canada S4S 0A2*
[b] *Department of Mechanical and Manufacturing Engineering, University of Calgary, Calgary, Alta., Canada T2N 1N4*

Received 10 September 2001; accepted 25 June 2002

Abstract

This paper is devoted to the development of a novel approximate and numerical method for the solutions of linear and non-linear oscillatory systems, which are common in engineering dynamics. The original physical information included in the governing equations of motion is mostly transferred into the approximate and numerical solutions. Therefore, the approximate and numerical solutions generated by the present method reflect more accurately the characteristics of the motion of the systems. Furthermore, the solutions derived are continuous everywhere with good accuracy and convergence in comparing with Runge–Kutta method. An approximate solution is developed for a linear oscillatory problem and compared with its corresponding exact solution. A non-linear oscillatory problem is also solved numerically and compared with the solutions of Runge–Kutta method. Both the graphical and numerical comparisons are provided in the paper. The accuracy of the approximate and numerical solutions can be controlled as desired by the number of terms in the Taylor series and the value of a single parameter used in the present work. Formulae for numerical computation in solving various linear and non-linear oscillatory problems by the new approach are provided in the paper.

1. Introduction

It is usual to express an oscillatory system by an ordinary differential equation in the form

$$\ddot{x} + 2c\dot{x} + \omega^2 x = f(t, x, \dot{x}), \tag{1}$$

where the coefficients c and ω are the physical properties of the system and $f(t, x, \dot{x})$ can be either a linear or a non-linear function of time, displacement and velocity of the oscillator considered. In

*Corresponding author. Tel.: +1-306-585-4498; fax: +1-306-585-4855.
E-mail address: liming.dai@uregina.ca (L. Dai).

PII: S0022-460X(02)01065-9

solving the above equation by the existing numerical methods such as Euler's method, Taylor series method and Runge–Kutta method [1,2], conventionally, the second order differential equation shown in Eq. (1) is transformed into a system of first order differential equations. On the basis of the first order differential equations, a recurrence relation for numerical calculation is then developed in the form

$$x_i = F(x_{i+1}, \dot{x}_{i+1}). \tag{2}$$

Obviously the numerical solution given by the above equation can only be discrete, and the solutions at the ith point, x_i, rely upon the initial conditions and the calculated solutions corresponding to the discrete points $i = 1, 2, 3, \ldots, i-1$. In addition, due to the mathematical operations such as linearization and Taylor expansion, the physical meaning implied in the original equation of motion is lost in the mathematical manipulations.

In this paper, a new approach namely the P-T method for approximately and numerically solving the linear and non-linear oscillatory problems is developed. The numerical results calculated by using the P-T method are presented and compared with the results generated by Runge–Kutta method which is probably the most popular numerical method in solving various linear and non-linear differential equations [2–4]. With the introduction of a piecewise-constant argument $[Nt]/N$, where $[Nt]$ represents the greatest-integer function and N an integer parameter, together with Taylor series expansion, the numerical solution of Eq. (1) is so developed that a linear oscillatory system is established between the two points $[Nt]/N$ and $([Nt]+1)/N$. Therefore, unlike the discrete solutions produced by the existing numerical methods, the approximate and numerical solutions produced by the P-T method are continuous everywhere on the entire time range from zero to t for any given value of N.

In numerically solving the oscillatory problems by the P-T method, the major portion of the corresponding original differential equations remains unchanged. Therefore, the solutions derived by the P-T method are more accurate than the existing numerical methods such as Runge–Kutta method. Most significantly, the P-T method reveals the actual physical behavior of the oscillatory systems to the maximum possible level in comparison with Runge–Kutta method and the other numerical methods with discrete solutions.

Since the numerical technique presented in this paper is based on a single-step method, and the step length for numerical calculations can be varied freely for each time interval, a step size control technique becomes possible.

2. Derivation of approximate and numerical solutions

Piecewise-constant arguments of form $[\cdot]$ have been widely used in the analysis on the delay differential equations [5,6]. To approximately or numerically solve non-linear dynamical problems, Dai and Singh [7,8] recently reported a piecewise-constant technique in which the piecewise-constant argument $[Nt]/N$ was introduced for solving the vibration problems. It was demonstrated in Refs. [7,8] that the following condition is satisfied as N approaches infinity:

$$\lim_{N\to\infty} \frac{[Nt]}{N} = t. \tag{3}$$

Consequently, the approximate solutions produced by a piecewise-constant technique employing the argument $[Nt]/N$ were proved to become the corresponding accurate solutions when N tended to infinity. In numerically solving a dynamical problem by the piecewise-constant technique, N was a chosen finite number and used as a factor to control the accuracy of the numerical results. The piecewise-constant technique produces reasonably accurate results. In order to further improve the efficiency of the numerical calculation based on the piecewise-constant technique, employment of Taylor series expansion seems to be a natural choice.

Taylor series expansion is essentially an expression, which can generally be used to approximate any function to any desired degree of accuracy. In considering this, a more accurate numerical solution may be anticipated if a linear or a non-linear governing equation can be replaced by a second order ordinary differential equation together with a power series of finite terms.

Making use of the piecewise-constant argument $[Nt]/N$ and Taylor series expansion of order n, Eq. (1) can be approximately expressed as follows:

$$\ddot{x}_i = g_i + g_i' t + \frac{1}{2!} g_i'' t^2 + \cdots + \frac{1}{n!} g_i^{(n)} t^n, \tag{4}$$

where $g = f(t, x, \dot{x}) - 2c\dot{x} - \omega^2 x$, and the subscript $i = 1, 2, 3, \ldots$ represents an arbitrary time interval of $[Nt]/N \leqslant t < ([Nt]+1)/N$, and further

$$g_i = g\left(\frac{[Nt]}{N}, x_i\left(\frac{[Nt]}{N}\right), \dot{x}_i\left(\frac{[Nt]}{N}\right)\right), \tag{5}$$

$$g_i' = \left[\frac{\mathrm{d}}{\mathrm{d}t} g(t, x_i, \dot{x}_i)\right]_{t=[Nt]/N}, \tag{6}$$

$$g_i'' = \left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} g(t, x_i, \dot{x}_i)\right]_{t=[Nt]/N}, \tag{7}$$

$$\vdots$$

$$g_i^{(n)} = \left[\frac{\mathrm{d}^n}{\mathrm{d}t^n} g(t, x_i, \dot{x}_i)\right]_{t=[Nt]/N}. \tag{8}$$

The solution to Eq. (4) is given by

$$x_i(t) = d_i + v_i t + \frac{1}{2!} g_i t^2 + \frac{1}{3!} g_i' t^3 + \cdots + \frac{1}{(n+2)!} g_i^{(n)} t^{n+2}. \tag{9}$$

This solution may be used for numerical purpose if it is expressed in the form

$$x_{i+1}(t) = d_i + v_i \frac{1}{N} + \frac{1}{2!} g_i \frac{1}{N^2} + \frac{1}{3} g_i' \frac{1}{N^3} + \cdots + \frac{1}{(n+2)!} g_i^{(n)} \frac{1}{N^{n+2}}, \tag{10}$$

where the displacement and velocity of the system at $t = [Nt]/N$ are given as

$$d_i = x_i\left(\frac{[Nt]}{N}\right), \quad v_i = \dot{x}_i\left(\frac{[Nt]}{N}\right). \tag{11}$$

The numerical solution calculated by Eq. (10) is similar in principle to that given by the numerical method of Taylor series of order n upon which the popular Runge–Kutta method is developed

[1,9]. The accuracy of the solution depends on the order n or the number of terms $n+1$ in Eq. (4). Making use of the piecewise-constant technique, the accuracy of the numerical calculation is, in addition, controlled by the parameter N.

A better numerical solution may be expected if the governing equation (1) is expressible in a form such that a portion of it is made analytically solvable. For instance, consider

$$\ddot{x} + 2c\dot{x} + \omega^2 x = \varphi(t) + \psi(t, x, \dot{x}). \tag{12}$$

In the above equation, the left-hand side together with the first term $\varphi(t)$ on the right-hand side is a non-homogeneous ordinary differential equation for which an analytical solution is readily available. In order to numerically solve this equation, $\psi(t, x, \ddot{x})$ is expressed as a function of t by Taylor series expansion, such that

$$\ddot{x}_i + 2c\dot{x}_i + \omega^2 x_i = \varphi_i(t) + \psi_i + \dot{\psi}_i t + \frac{1}{2!}\ddot{\psi}_i t^2 + \cdots + \frac{1}{n!}\psi_i^{(n)} t^n, \tag{13}$$

which is valid on an arbitrarily small ith time interval, $[Nt]/N \leqslant t < ([Nt]+1)/N$. In comparison with Eq. (4), the above equation contains much more of the original physical information embedded in the governing equation (1), it is likely to have a solution closer to the accurate solution to Eq. (1).

For a numerical technique, it is important to ensure that the convergence of the numerical solution with the original dynamical system is satisfied. To analyze the convergence of the numerical solutions derived through the present technique and the truncation error caused by the P-T method, the governing equation of the oscillatory problem in a general form of Eq. (12) can be considered. With the P-T method, the solution of this system within the interval $[Nt]/N \leqslant t < ([Nt]+1)/N$, can be obtained by the following governing equation:

$$\ddot{x}_i + 2c\dot{x}_i + \omega^2 x_i = \varphi(t) + \psi\left(\frac{[Nt]}{N}, x_i\left(\frac{[Nt]}{N}\right), \dot{x}_i\left(\frac{[Nt]}{N}\right)\right). \tag{14}$$

Thus, within this interval, the difference between the continuous system with solution $x(t)$ to Eq. (12), and the system governed by this equation of piecewise-constant system is expressible as

$$\ddot{x} + 2c\dot{x} + \omega^2 x - \varphi(t) - \psi\left(\frac{[Nt]}{N}, x_i\left(\frac{[Nt]}{N}\right), \dot{x}_i\left(\frac{[Nt]}{N}\right)\right) = R_N. \tag{15}$$

Usually, the truncation error R_N is not zero. Employing Taylor expansion, the derivatives shown in the above equation can be given as

$$\ddot{x}(t) = \ddot{x}\left(\frac{[Nt]}{N}\right) + x'''\left(\frac{[Nt]}{N}\right)\left(t - \frac{[Nt]}{N}\right) + 0\left(\left(t - \frac{[Nt]}{N}\right)^2\right), \tag{16}$$

$$\dot{x}(t) = \dot{x}\left(\frac{[Nt]}{N}\right) + \ddot{x}\left(\frac{[Nt]}{N}\right)\left(t - \frac{[Nt]}{N}\right) + 0\left(\left(t - \frac{[Nt]}{N}\right)^2\right) \tag{17}$$

and

$$x(t) = x\left(\frac{[Nt]}{N}\right) + \dot{x}\left(\frac{[Nt]}{N}\right)\left(t - \frac{[Nt]}{N}\right) + 0\left(\left(t - \frac{[Nt]}{N}\right)^2\right). \tag{18}$$

From Eq. (12),

$$\ddot{x}\left(\frac{[Nt]}{N}\right) + 2c\dot{x}\left(\frac{[Nt]}{N}\right) + \omega^2 x\left(\frac{[Nt]}{N}\right) - \phi\left(\frac{[Nt]}{N}\right) - \psi\left(\frac{[Nt]}{N}, x\left(\frac{[Nt]}{N}\right), \dot{x}\left(\frac{[Nt]}{N}\right)\right) = 0. \quad (19)$$

Thus,

$$R_N = \left[x'''\left(\frac{[Nt]}{N}\right) + 2c\ddot{x}\left(\frac{[Nt]}{N}\right) + \omega^2 \dot{x}\left(\frac{[Nt]}{N}\right)\right]\left(t - \frac{[Nt]}{N}\right) + 0\left(\left(\frac{1}{N}\right)^2\right), \quad (20)$$

whereas $|t - [Nt]/N| \leqslant 1/N$.

Utilizing the conclusion indicated in Eq. (3) and the proof in Ref. [8], the truncation in the above equation is zero as $N \to \infty$ and the time integral step tends to zero, such that the difference between the exact solution and the numerical solution is vanished correspondingly.

In the proof above, c and ω are bounded, as $N \to \infty$, $\ddot{x}([Nt]/N) \to \ddot{x}(t)$, $\dot{x}([Nt]/N) \to \dot{x}(t)$, $x([Nt]/N) \to x(t)$, $\varphi([Nt]/N) \to \varphi(t)$, and $\psi([Nt]/N, x([Nt]/N), \dot{x}([Nt]/N)) \to \psi(t, x, \dot{x})$, hence, the piecewise-constant system becomes the cotinuous system.

3. Solution of a linear system

To elucidate the above point more clearly, consider the following equation of motion representing a damped linear oscillatory system:

$$\ddot{x}(t) + 2c\dot{x}(t) + a^2 x(t) = bx(t) \quad (21)$$

for which an analytical solution is available. Based on the discussion above, this equation is replaced by the following equation with Taylor expansion for the term $bx(t)$:

$$\ddot{x}_i(t) + 2c\dot{x}_i(t) + a^2 x_i(t) = bd_i + bv_i\left(t - \frac{[Nt]}{N}\right), \quad (22)$$

which is valid on the ith interval $[Nt]/N \leqslant t < ([Nt]+1)/N$. In this equation, only the first two terms of the Taylor series expansion are considered and the rest of the higher order terms are neglected.

Solution of Eq. (22) is expressible as

$$x_i = \mathrm{e}^{-c\left(t-[Nt]/N\right)}\left\{B_1 \cos\left[\xi\left(t - \frac{[Nt]}{N}\right)\right] + B_2 \sin\left[\xi\left(t - \frac{[Nt]}{N}\right)\right]\right\} + A_1 + A_2\left(t - \frac{[Nt]}{N}\right) \quad (23)$$

in which

$$\xi^2 = a^2 - c^2, \quad (24)$$

$$A_2 = \frac{1}{a^2}, \quad A_1 = \frac{1}{a^2}(bd_i - 2cA_2), \quad (25)$$

$$B_1 = d_1 - A_1, \quad B_2 = \frac{1}{\xi}(v_i + cB_1 - A_2). \quad (26)$$

It may be noted in Eq. (23) that the displacement $x(t)$ and velocity $\dot{x}(t)$ are continuous in the time interval. Because of the continuity of x and $\dot{x}$ on $t \in [0, \infty)$, the following conditions must be satisfied:

$$x_i\left(\frac{[Nt]}{N}\right) = x_{i-1}\left(\frac{[Nt]}{N}\right) \quad \text{and} \quad \dot{x}_i\left(\frac{[Nt]}{N}\right) = \dot{x}_{i-1}\left(\frac{[Nt]}{N}\right). \tag{27}$$

The above conditions of continuity lead to the recurrence relations

$$d_{i+1} = \mathrm{e}^{-c/N}\left(B_1 \cos\frac{\xi}{N} + B_2 \sin\frac{\xi}{N}\right) + A_1 + \frac{A_2}{N}, \tag{28}$$

$$v_{i+1} = -c\mathrm{e}^{-c/N}\left(B_1 \cos\frac{\xi}{N} + B_2 \sin\frac{\xi}{N}\right) + \mathrm{e}^{-c/N}\left(-\xi B_1 \sin\frac{\xi}{N} + \xi B_2 \cos\frac{\xi}{N}\right) + A_2. \tag{29}$$

With Eq. (23) and recurrence relations (28) and (29), a numerical solution can be obtained through a step-by-step procedure.

In order to compare the accuracy of this solution with that of the results calculated by using the existing numerical methods such as Taylor series and Runge–Kutta methods, the solution given by Eq. (23) is expanded by Taylor series into the power series form

$$\begin{aligned} x_i = {} & d_i + v_i\left(t - \frac{[Nt]}{N}\right) - \left[(a^2-b)\frac{d_i}{2} + cv_i\right]\left(t - \frac{[Nt]}{N}\right)^2 \\ & + \left[\frac{cd_i}{3}(a^2-b) - \frac{v_i}{6}(a^2-b-4c^2)\right]\left(t - \frac{[Nt]}{N}\right)^3 \\ & + \left[\frac{d_i}{24}(a^2-b)(a^2-4c^2) + \frac{cv_i}{6}\left(a^2 - \frac{b}{2} - 2c^2\right)\right]\left(t - \frac{[Nt]}{N}\right)^4 + \cdots. \end{aligned} \tag{30}$$

The exact solution for Eq. (21) has a closed form [10]. Employing the piecewise-constant argument $[Nt]/N$, the exact solution of Eq. (21) can also be expressed into a power series form on the ith time interval, such as

$$\begin{aligned} x_i = {} & d_i + v_i\left(t - \frac{[Nt]}{N}\right) - \left[(a^2-b)\frac{d_i}{2} + cv_i\right]\left(t - \frac{[Nt]}{N}\right)^2 \\ & + \left[\frac{cd_i}{3}(a^2-b) - \frac{v_i}{6}(a^2-b-4c^2)\right]\left(t - \frac{[Nt]}{N}\right)^3 \\ & + \left[\frac{d_i}{24}(a^2-b)(a^2-b-4c^2) + \frac{cv_i}{6}(a^2-b-2c^2)\right]\left(t - \frac{[Nt]}{N}\right)^4 \\ & + \frac{1}{120}\{-4c(a^2-b)(a^2-b-2c^2)d_i \\ & \quad + [(a^2-b)^2 - 4c^2(3a^2-3b-4c^2)]v_i\}\left(t - \frac{[Nt]}{N}\right)^5 + \cdots. \end{aligned} \tag{31}$$

It can be seen from Eqs. (4) and (9) that the first two terms on the right-hand side of Eq. (4) are necessary for obtaining the solution of third order in Eq. (10), which is developed by the method of direct Taylor series expansion [1]. All the other terms in Eq. (10) that are higher than the third orders are neglected. The numerical solution given by Eq. (23) is a closed-form solution to Eq. (22) that also retains two terms of Taylor series expansion. However, in contrast to the solution shown in Eq. (10), all the higher order terms are still available in the solution given by Eq. (30) which is based on Eq. (23). The first four terms in Eq. (30) are identical to those of the exact solution presented in Eq. (31), hence, the accuracy equivalent to that of the Taylor series method of order three is ensured. Significantly, the other higher order terms appearing in Eq. (30) are very close to the corresponding terms in Eq. (31) of the exact solution. Therefore, the numerical solution generated by Eq. (30) must be a better approximation to the exact solution. In comparing with the exact solution in Eq. (31), the difference between the solution provided by direct Taylor series expansion in Eq. (10) and the solution provided by Eq. (30) is significant, and the difference is independent of the parameter N.

It is clear, if the higher order terms of the Taylor expansion for the term bx are further considered in Eq. (22), that a more accurate numerical solution may be obtained. Through the same procedure as discussed above, a numerical solution to the following equation of motion containing one more higher order term can be derived:

$$\ddot{x}(t) + 2c\dot{x}(t) + a^2x(t) = bd_i + bv_i\left(t - \frac{[Nt]}{N}\right) + \frac{b}{2!}(bd_i - 2cv_i - a^2d_i)\left(t - \frac{[Nt]}{N}\right)^2. \tag{32}$$

This equation has a theoretical solution in the closed form similar to Eq. (23). Also, the solution for Eq. (32) is continuous and can be expressed in the following Taylor series expansion form for the convenience of comparison with the exact solution:

$$\begin{aligned} x_i = {} & d_i + v_i\left(t - \frac{[Nt]}{N}\right) - \left[(a^2 - b)\frac{d_i}{2} + cv_i\right]\left(t - \frac{[Nt]}{N}\right)^2 \\ & + \left(\frac{cd_i}{3}(a^2 - b) - \frac{v_i}{6}(a^2 - b - 4c^2)\right)\left(t - \frac{[Nt]}{N}\right)^3 \\ & + \left[\frac{d_i}{24}(a^2 - b)(a^2 - b - 4c^2) + \frac{cv_i}{6}(a^2 - b - 2c^2)\right]\left(t - \frac{[Nt]}{N}\right)^4 \\ & + \frac{1}{120}\left\{ - 4c(a^2 - b)\left(a^2 - \frac{b}{2} - 2c^2\right)d_i \right. \\ & \left. \qquad + [a^2(a^2 - b) - 4c^2(3a^2 - 2b - 4c^2)]v_i \right\}\left(t - \frac{[Nt]}{N}\right)^5 + \cdots. \end{aligned} \tag{33}$$

In comparing with the exact solution in Eq. (31), this solution corresponding to the governing equation (32) has an accuracy equivalent to that of the Taylor series method of order four in terms of displacement x, or of order three in terms of velocity $\dot{x}$. Again, the coefficients in the fifth order term in Eq. (33) are very close to the corresponding terms in the exact solution of Eq. (31). Eqs. (30) and (31) are actually the continuous approximate solutions to the governing Eq. (21). For numerical calculation, the recurence relations corrsponding to Eq. (32) can be determined by its solution together with the conditions of continuity demonstrated in Eq. (27). Also, it can be shown that the solution with an accuracy of order four in velocity can be developed in a similar manner if the third order term

$$\frac{b}{3!}[2c(a^2 - b)d_i - (a^2 - b - 4c^2)v_i]\left(t - \frac{[Nt]}{N}\right)^3 \tag{34}$$

is added on the right-hand side of the equal sign in Eq. (32).

As can be seen from the above discussion, the main idea of the present approach in solving oscillatory problems is to maintain as much as possible the original physical information of the governing equation. The piecewise-constant procedure applied as less as possible to the terms in the governing equation allows a greater portion of the original information to be kept intact in the numerical calculation. In return, a higher accuracy solution is developed. The piecewise-constant argument $[Nt]/N$, which has the obvious advantage of bridging the continuous and piecewise-constant systems, is also a contribution to simplifying the procedure of deriving the numerical solutions discussed above. One may note that the numerical method discussed above is actually a technique combining the idea of piecewise-constant technique [7,8] and Taylor series method. It is therefore called the P-T method hereafter.

With the approximate solutions provided by the P-T method, numerical solutions can be easily generated by the recurrent relations developed via the approximate solutions and the conditions of continuity. The Runge–Kutta method of fourth order is probably the most popular numerical method in solving the oscillatory problems, and it is a method better than the other numerical methods such as Taylor's expansion and Euler's method for solving the oscillatory problems [1]. In order to compare the results produced by the P-T method with those generated by the existing numerical methods in analyzing oscillatory problems, both the P-T method and Runge–Kutta method with the identical order of accuracy are used for solving an oscillatory system governed by the equation

$$\ddot{x} + 0.2\dot{x} + 10x = x. \tag{35}$$

Eq. (35) is linear and its exact solution is readily available for comparison. One may probably note that the Runge–Kutta method provides numerical solutions at discrete points, therefore, a continuous solution in Taylor series expansion form as those shown in Eqs. (30) and (31) is not available.

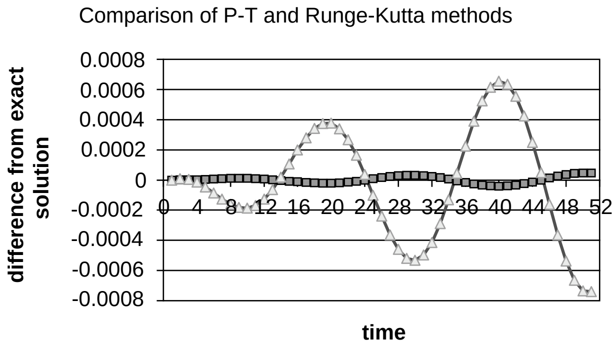

Fig. 1. Comparison of numerical results generated by the P-T method of fourth order and Runge–Kutta method of fourth order for the governing equation $\ddot{x} + 0.2\dot{x} + 10x = x$, with $x(0) = 1$ and $\dot{x}(0) = 2$. The curve —△— denotes the error of the P-T method and the curve —■— represents the error of the Runge–Kutta method.

Table 1
Comparison of the CPU times for solving $\ddot{x}+0.2\dot{x}+10x=x$. For both the methods of P-T and Runge–Kutta of fourth order, the initial conditions are $x(0)=1$ and $\dot{x}(0)=2$; with a time range from 0.0 to 20 000; and step length 0.3

Method	CPU time (s)
P-T	8.31
Runge–Kutta	8.33

Fig. 1 exhibits the numerical solutions for Eq. (35) plotted with the numerical data generated by both the P-T method and Runge–Kutta method of the fourth order. The vertical axis denotes the difference between a numerical solution and the exact solution of Eq. (35), the horizontal axis represents the time, and the two curves represent the errors of the P-T method of fourth order and Runge–Kutta method of fourth order, respectively. It can be visualized from Fig. 1; the fourth order P-T method supplies a more accurate numerical solution in comparison with the Runge–Kutta method of order four.

It should be noted that the exact solution used for the figure is precisely the analytical solution of Eq. (35). The errors of the numerical solutions obtained through the P-T and Runge–Kutta methods are calculated with respect to the exact solution. The errors of the numerical solutions are caused by the local truncation errors and the roundoff errors due to the finite-digit arithmetic of the computing system as most of the other numerical methods.

The speed of numerical calculation of the P-T method is slightly faster than that of the Runge–Kutta method. To compare the speed of the two methods, the CPU times of the actual calculations for solving Eq. (35) are tabulated in Table 1.

4. Solution of a non-linear system

Evidently, the P-T method can also be applied to non-linear dynamical systems in providing approximate or numerical solutions. The following governing equation is typical for a non-linear or linear oscillatory system:

$$\ddot{x}(t)+2c\dot{x}(t)+a^2x(t)=f(x,\dot{x},t). \tag{36}$$

By the P-T method, expand function $f(x,\dot{x},t)$with Taylor series to the desired order of accuracy on an ith time interval $[Nt]/N\leqslant t<([Nt]+1)/N$, such that

$$\ddot{x}_i(t)+2c\dot{x}_i(t)+a^2x_i(t)=f_{[Nt]/N}+f'_{[Nt]/N}\left(t-\frac{[Nt]}{N}\right)+\frac{1}{2!}f''_{[Nt]/N}\left(t-\frac{[Nt]}{N}\right)^2+\frac{1}{3!}f'''_{[Nt]/N}\left(t-\frac{[Nt]}{N}\right)^3+\cdots. \tag{37}$$

A complete solution for Eq. (37) is then readily available with the desired order of accuracy, and the recurrence relation for numerical calculation can be consequently developed through the procedure as demonstrated previously for the linear system.

In order to have an approximate solution with the same accuracy as that of the Runge–Kutta method of fourth order, employ the first four terms in Eq. (37) for the function $f(x, \dot{x}, t)$ and truncate the higher order terms. The approximate solution in $[Nt]/N \leqslant t < ([Nt]+1)/N$ can be developed for Eq. (37) as

$$x_i = \mathrm{e}^{-c(t-[Nt]/N)}\left\{B_1 \cos\left[\xi\left(t - \frac{[Nt]}{N}\right)\right] + B_2 \sin\left[\xi\left(t - \frac{[Nt]}{N}\right)\right]\right\} + A_1 + A_2\left(t - \frac{[Nt]}{N}\right) + A_3\left(t - \frac{[Nt]}{N}\right)^2 + A_4\left(t - \frac{[Nt]}{N}\right)^3, \tag{38}$$

where

$$\xi^2 = a^2 - c^2, \quad A_4 = \frac{1}{6a^2} f'''_{[Nt]/N}, \tag{39,40}$$

$$A_3 = \frac{1}{a^2}\left(\tfrac{1}{2} f''_{[Nt]/N} - 6cA_4\right), \quad A_2 = \frac{1}{a^2}(f'_{[Nt]/N} - 4cA_3 - 6A_4), \tag{41,42}$$

$$A_1 = \frac{1}{a^2}(f_{[Nt]/N} - 2cA_2 - 2A_3), \tag{43}$$

$$B_1 = d_i - A_1, \quad B_2 = \frac{1}{\xi}(v_i + cB_1 - A_2). \tag{44}$$

It should be noted that the above formulae developed for the P-T method are suitable for solving any linear or non-linear dynamic problems in a general form as exhibited in Eq. (36).

To compare the numerical results of the P-T method and Runge–Kutta method, a non-linear system governed by the following equation is considered.

$$\ddot{x}(t) + 2c\dot{x}(t) + a^2 x(t) = bx^3. \tag{45}$$

The recurrence relations of the P-T method for numerical solution of this non-linear system can be generated by simply substituting $f(x) = bx^3$ into Eqs. (36) and (37) with the following differentiations of $f(x)$ together with the conditions of continuity demonstrated in Eq. (27), such that

$$f_{[Nt]/N} = bd_i^3, \quad f'_{[Nt]/N} = 3bd_i^2 v_i, \tag{46}$$

$$f''_{[Nt]/N} = 6bd_i v_i^2 + 3bd_i^2 G_1, \quad \text{where } G_1 = bd_i^3 - 2cv_i - a^2 d_i, \tag{47}$$

$$f'''_{[Nt]/N} = 6bd_i v_i^3 + 18bd_i v_i G_1 + 3bd_i^2 v_i G_2, \quad \text{where } G_2 = f'_{[Nt]/N} - 2cG_1 - a^2 v_i. \tag{48}$$

Fig. 2 exhibits the comparison of the numerical solutions for an example calculated by both the P-T method and Runge–Kutta method. Numerical solution by the Runge–Kutta method for the governing equation (45) is obtained by the conventional Runge–Kutta method of fourth order to compare with the solution by the P-T method of the fourth order. In Fig. 2, the dashed line and the line conneced with small triangles are the curves representing the solutions by the P-T method

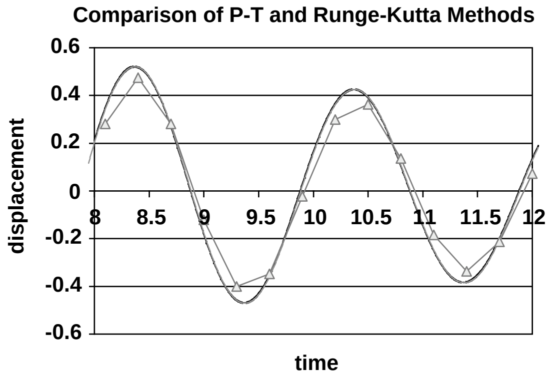

Fig. 2. Comparison of numerical results generated by the P-T method of fourth order and Runge–Kutta method of fourth order for the governing equation $\ddot{x} + 0.2\dot{x} + 10x = x^3$, with $x(0) = 1$ and $\dot{x}(0) = 2$. In the figure, the dashed line represents the P-T method with step length 0.3; the —△— line represents the Runge–Kutta method with step length 0.3 and the solid line designates the Runge–Kutta method with step length 0.003.

and Runge–Kutta method, respectively. The step length for the numerical calculations of both the solutions of the P-T and Runge–Kutta methods is kept to be 0.3.

As can be seen from the figure, and also as discussed previously, the solution of Runge–Kutta method is discrete whereas the solution of the P-T method is continuous and smooth everywhere. Combining the continuous solutions within the time intervals and the continuity conditions shown in Eq. (27), the dashed line is actually continuous and smooth in the entire time range considered. In fact, for each time interval of a chosen step length, there is a continuous oscillatory solution corresponding to it. The oscillatory behavior of the system is governed by Eq. (38) in which the original physical information is maintained and protected to the utmost level.

It is well known that a smaller step length may yield a more accurate numerical solution [1,2]. Employing Runge–Kutta method of fourth order, with a much smaller step length of 0.003 unit, a more accurate numerical solution is obtained and represented by the solid line in Fig. 2. This solution can be considered as the one with an accuracy of very high level in comparison to the solution with a step length a hundred times larger. As can be observed from the figure, the solution of the P-T method with step length 0.3 matches very well with that of the Runge–Kutta method with step length 0.003, the two curves are overlapped almost everywhere. On the other hand, the Runge–Kutta method with step length 0.3 provides a discrete solution with significantly lower accuracy in comparison with that of the P-T method of the same step length. Clearly, the P-T method provides a solution with higher accuracy in comparison with the Runge–Kutta method. In addition, the solution calculated by the P-T method is continuous in the entire time range considered.

Although the solid curve seems smooth, the curve is in fact formed by short straight lines connecting the end points at $t = [Nt]/N$ and $t = ([Nt] + 1)/N$, no matter how small the step length is taken. It is also significant to note that the curve segment generated by the P-T method, corresponding to a step length of 0.3 unit, yielded by the solution of Eq. (38) that reflects the physical characteristics of the oscillatory system, has the same shape as the corresponding curve segment of the solid curve of the Runge–Kutta method with higher accuracy. Evidently, the

Table 2
Accuracy comparisons for forth order P-T method and Runge–Kutta method of the same order in solving for $\ddot{x}(t) + 2c\dot{x}(t) + a^2x(t) = bx^3$. The initial conditions are $x(0) = 1$ and $\dot{x}(0) = 2$

Time	P-T Step 0.3	Runge–Kutta Step 0.3	Runge–Kutta Step 0.003
8.1	0.347724	0.281461	0.343268
8.4	0.517027	0.473953	0.518208
8.7	0.267069	0.28183	0.272421
9.0	−0.18254	−0.1204	−0.17695
9.3	−0.45882	−0.40096	−0.45817
9.6	−0.35455	−0.34717	−0.35839
9.9	0.027678	−0.02265	0.021947
10.2	0.364802	0.298642	0.362458
10.5	0.390918	0.362595	0.393205
10.8	0.1028	0.135726	0.107953
11.1	−0.25117	−0.18411	−0.24755
11.4	−0.3827	−0.33687	−0.38347
11.7	−0.20041	−0.2129	−0.20458
12	0.132724	0.072054	0.128444

Table 3
Comparison of the CPU times for solving $\ddot{x} + 0.2\dot{x} + 10x = x^3$. For both the methods of P-T and Runge–Kutta of fourth order, the initial conditions are $x(0) = 1$ and $\dot{x}(0) = 2$; time range from 0.0 to 20 000; and step length 0.3

Method	CPU time (s)
P-T	8.36
Runge–Kutta	8.37

solution by the P-T method is a good approximation to the motion of the governing equation. Solution of the P-T method not only generates a continuous curve, as against a straight line to connect the two points of a time interval, but also provides a curve that represents the actual physical behavior of the original non-linear oscillatory system.

Even though the difference between the dashed curve and the solid curve is hardly distinguishable by naked eyes, the numerical values corresponding to the two curves are actually different. To demonstrate the difference between the two curves, the detailed numerical data calculated by the P-T and Runge–Kutta methods corresponding to the curves are listed in Table 2. From Fig. 2 and the table, one may again conclude that the solution given by the P-T method is more accurate in comparison with Runge–Kutta method as expected. The high accuracy of the P-T method evidently is related to the advantage that the maximum possible original physical information is kept from variation when developing the equations for the approximate and numerical analysis via the P-T method.

The CPU times spent for the calculations employing both the P-T method and the Runge–Kutta method are shown in Table 3.

Note that the CPU time of 8.37 s shown in Table 3 for the Runge–Kutta method is obtained with step length of 0.3. The CPU time will be enormously longer than 8.37 s for the Runge–Kutta method if the step length of 0.003 is considered.

5. Conclusions

It can be seen from the above discussion that the P-T method is an efficient approximate and numerical technique, which provides sufficiently accurate results with good convergence for both the linear and non-linear oscillatory problems. In comparison with the Runge–Kutta method and the other existing approximate and numerical methods, the following characteristics of the approximate and numerical computations with the P-T method need to be stated.

The approximate solutions derived by the P-T method are continuous in the length-adjustable interval $[Nt]/N \leqslant t < ([Nt]+1)/N$ and the entire time range for $t > 0$. Theoretically, in the case of linear systems, the difference between the approximate solution produced by the P-T method and the exact solution will vanish as the parameter N of the piecewise-constant argument $[Nt]/N$ approaches infinity.

Most existing numerical methods provide the solutions only at discrete points. The solution corresponding to the time interval in between the discrete points is usually not available. In contrast to these numerical methods, the numerical solution derived by the P-T method, such as the solutions shown in Eqs. (23), (33) and (38) are continuous everywhere in the entire time range for any finite value of N. Owing to the properties of the piecewise-constant argument $[Nt]/N$, the equation of the system and the corresponding recurrence relations can be numerically solved for the desired time range.

In numerically solving the oscillatory problems, usually, a second order differential equation has to be transformed into a system of two first order differential equations. Numerical solutions corresponding to the first order differential equations are then developed by employing mathematical operations such as linearization or Taylor expansion [1,11]. In doing so, the physical meaning implied in the original equation of motion is quite often lost in the manipulations of the pure mathematical expressions. However, the P-T method attempts to keep the form of the equation and the physical information involved in the original governing equation unchanged as much as possible during the process of the approximations or numerical calculations. As a result, a solution generated by the P-T method makes much more physical sense in dynamic systems. This is significant especially when dynamical systems of high non-linearity are considered. In addition, the P-T method gives a higher accuracy in comparison with the existing numerical methods. Due to the maintenance of the original physical information and the piecewise argument, the continuous solution given by the P-T method can also be used as an approximate solution to the exact or accurate solution on the time interval and over the time range desired. The accuracy of the solution is controlled by the number of terms expanded by Taylor series and the value of N.

The P-T method and Runge–Kutta method both rely on the Taylor expansion [11]. Similar to the Runge–Kutta method, the P-T method can be applied to solve systems of differential equations or the systems of multi-degrees of freedom. In fact, the P-T method to solve systems of difference equations is the generalization of the method for the single differential equation as presented previously.

Iteration is a major operation for the numerical calculations of many numerical methods. When the local initial conditions are given, the iteration must be repeatedly carried out to obtain the numerical solution at the end of the time interval. However, there is no iteration involved in the numerical calculations by using the P-T method. Once the local initial conditions are available, the solution at the end of the time interval and at any point within the time interval can be directly calculated by the formulae derived with employment of the P-T method.

The numerical technique presented is based on a single-step method with freely variable step length; a step size control technique can be easily applied to further increase the accuracy and efficiency of the numerical calculations with the P-T method.

References

[1] S. Nakamura, Applied Numerical Methods with Software, Prentice-Hall, Englewood Cliffs, NJ, 1991.

[2] D.W. Zingg, T.T. Chisholm, Runge–Kutta methods for linear ordinary differential equations, Applied Numerical Mathematics 31 (1999) 227–238.

[3] M.I. Abukhaled, E.J. Allen, A class of second-order Runge–Kutta methods for numerical solution of stochastic differential equations, Stochastic Analysis and Applications 16 (1998) 977–992.

[4] A. Iserles, G. Ramaswami, M. Sofroniou, Runge–Kutta methods for quadratic ordinary differential equations, BIT—Numerical Mathematics 38 (1998) 315–336.

[5] K.N. Jayasree, S.G. Deo, On piecewise constant delay differential equations, Journal of Mathematical Analysis and Applications 169 (1992) 55–69.

[6] K.L. Cooke, J. Wiener, Stability regions for linear equations with piecewise continuous delay, Computers and Mathematics with Applications 12A (1986) 695–701.

[7] L. Dai, M.C. Singh, On oscillatory motion of spring–mass systems subjected to piecewise constant forces, Journal of Sound and Vibration 173 (1994) 217–233.

[8] L. Dai, M.C. Singh, An analytical and numerical method for solving linear and nonlinear vibration problems, International Journal of Solids and Structures 34 (1997) 2709–2731.

[9] L. Dai, M.C. Singh, Periodic, quasiperiodic and chaotic behavior of a driven Froude pendulum, International Journal of Nonlinear Mechanics 33 (1998) 947–965.

[10] W. Weaver Jr., S. Timoshenko, D.H. Young, Vibration Problems in Engineering, Wiley, New York, 1990.

[11] M. Friedman, A. Kandel, Fundamentals of Numerical Analysis, CRC Press, London, 1994.